Reading seminar on functor calculus — talk 1

Historical and algebraic calculus

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Abstract

These are the notes from a talk I gave at the Münster Functor Calculus Seminar on 25 February 2015. If you spot any mistakes in these notes, let me know, and I'll update them.

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Outline

(1) *Motivation*: polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$
(2) *The work of Eilenberg and Mac Lane [EML54]*: cross-effects for functors $\mathsf{Ab} \to \mathsf{Ab}$, applications to computing $H_*(K(\pi, n); \mathbb{Z})$ in low degrees.
(3) *Hartl-Vespa [HV11]*: a generalisation to source categories $\mathcal{C}$ with finite coproducts and a null object; the Taylor tower of a functor.
(4) *Hartl-Pirashvili-Vespa [HPV12]*: a further generalisation to source categories $\mathcal{C}$ with a monoidal product whose unit object I is null.
(5) *Djament-Vespa [DV13]*: inductive definition of degree; extension to the case where I is initial but not terminal; discussion of examples related to $H_*(IA_n)$.

1 Polynomial functions

In this section we'll consider smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ which are *reduced*, meaning that $f(0) = 0$. To begin with we define the *cross-effects* of such functions.

Definition 1.1 (Cross-effects) We define the cross-effects $cr_k f$ recursively by

$$\begin{aligned} cr_1 f &= f \\ cr_2 f(a_1, a_2) &= f(a_1 + a_2) - f(a_1) - f(a_2) \\ cr_n f(a_1, \ldots, a_n) &= cr_2(cr_{n-1}(-, a_3, \ldots, a_n))(a_1, a_2). \end{aligned}$$

Note that each $cr_n f$ is *multireduced* (its value is zero whenever any of its inputs is zero) and symmetric. By induction one can prove that

$$f(a_1 + \cdots + a_n) = \sum_{k=1}^{n} \sum_{j_1 < \cdots < j_k} cr_k f(a_{j_1}, \ldots, a_{j_k}). \tag{1.1}$$

Corollary 1.2 *If f is polynomial of degree d, the values $cr_1 f(1), cr_2(1,1), \ldots, cr_d(1, \ldots, 1) \in \mathbb{R}$ determine f uniquely.*

Proof. We know that $f(0) = 0$, and by (1.1) we know the values of $f(1), \dots, f(d)$. Note that a polynomial is determined by its values on $d+1$ points. $\square$

Another fact that one can check is that the nth cross-effect $cr_n f$ determines the derivative of f at 0:

$$\lim_{a_i \to 0} \left(\frac{cr_n f(a_1, \dots, a_n)}{a_1 \cdots a_n} \right) = f^{(n)}(0). \tag{1.2}$$

Corollary 1.3 *If f is analytic and $cr_{d+1} f \equiv 0$ then f is polynomial of degree at most d.*

Proof. Let $r > 0$ be the radius of convergence of f at 0. For $|x| < r$ we have the Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Since $cr_{d+1} f \equiv 0$, we also have $cr_n f \equiv 0$ for all $n > d$, and so by (1.2) we have $f^{(n)}(0) = 0$ for $n > d$. Hence f is a polynomial on $(-r, r)$. As Hironori pointed out, this immediately implies that f is polynomial globally, i.e. on all of $\mathbb{R}$, since analytic functions which agree on an open set are automatically equal. In the talk I gave an explicit proof that $f|_{[-\ell,\ell]}$ is polynomial for all $\ell > 0$ (implying that f is polynomial on $\mathbb{R}$), which effectively consisted in reproving this fact about analytic functions. $\square$

Hence for analytic functions $f \colon \mathbb{R} \to \mathbb{R}$ we may *define* "polynomial" to mean that $cr_{d+1} f$ vanishes for some d, and the degree of f is then the minimal such d.

2 Computing homology of Eilenberg-MacLane spaces

In [EML54], the setting is that we consider functors $T \colon \mathsf{Ab} \to \mathsf{Ab}$ such that $T(0) = 0$ (reduced). To define the cross-effects, we first define certain idempotents. Let $n \geqslant 1$ and $S \subseteq \{1, \dots, n\}$ and choose abelian groups $A_1, \dots, A_n$. There is an endomorphism

$$e_S(A_1, \dots, A_n) \colon \textstyle\bigoplus_i A_i \to \bigoplus_i A_i$$

defined to be the identity on components corresponding to elements $i \in S$ and the zero morphism otherwise. Note that this is an idempotent, and therefore so is its image $T(e_S(A_1, \dots, A_n))$. Taking an alternating sum, we obtain an endomorphism

$$\sum_{S \subseteq \{1,\dots,n\}} (-1)^{n-|S|}\, T(e_S(A_1, \dots, A_n)) \colon T(\textstyle\bigoplus_i A_i) \longrightarrow T(\bigoplus_i A_i).$$

Definition 2.1 The functor $cr_n T \colon \mathsf{Ab}^n \to \mathsf{Ab}$ is defined on objects as follows: $cr_n T(A_1, \dots, A_n)$ is the image of the above endomorphism. Given a morphism $(f_1, \dots, f_n)$ in Ab^n, this induces a morphism $T(\bigoplus_i f_i) \colon T(\bigoplus_i A_i) \to T(\bigoplus_i B_i)$, whose restriction to $cr_n T(A_1, \dots, A_n) \subseteq T(\bigoplus_i A_i)$ defines $cr_n T(f_1, \dots, f_n)$.

For example when $n = 2$ we have

$$cr_2 T(A_1, A_2) = \mathrm{image}(T(\mathrm{id} = \mathrm{pr}_1 + \mathrm{pr}_2) - T(\mathrm{pr}_1) - T(\mathrm{pr}_2)),$$

where $\mathrm{pr}_1, \mathrm{pr}_2 \colon A_1 \oplus A_2 \longrightarrow A_1 \oplus A_2$ set the second and first coordinates to zero respectively. From this one can see a very strong formal analogy with the cross-effects of the previous section. The first important fact about cross-effects is as follows:

Theorem 2.2 ([EML54, Theorem 9.1]) *There is a natural isomorphism*

$$T\Big(\bigoplus_{i=1}^{n} A_i\Big) \quad \cong \bigoplus_{\{s_1,\dots,s_k\} \subseteq \{1,\dots,n\}} cr_k T(A_{s_1}, \dots, A_{s_k}) \tag{2.1}$$

of functors $\mathsf{Ab}^n \to \mathsf{Ab}$.

Some remarks about the cross-effects:

- Each cr_kT is multireduced and symmetric up to natural isomorphism.
- The collection $\{cr_kT\}$ is unique with respect to the above properties, in the following sense. Suppose $U_S \colon \mathsf{Ab}^{|S|} \to \mathsf{Ab}$ is a collection of multireduced functors indexed by subsets $S \subseteq \{1,\ldots,n\}$ such that there is a natural isomorphism $T(\bigoplus_i A_i) \cong \bigoplus_{S\subseteq\{1,\ldots,n\}} U_S(A_{s_1},\ldots,A_{s_k})$, where $S = \{s_1,\ldots,s_k\}$ and $s_1 < \cdots < s_k$, then there are natural isomorphisms $U_S \cong cr_{|S|}T$.
- Natural transformations $T \to T'$ induce natural transformations $cr_kT \to cr_kT'$, so the kth cross-effect is a functor
$$cr_k \colon \mathsf{Fun}(\mathsf{Ab},\mathsf{Ab}) \longrightarrow \mathsf{Fun}(\mathsf{Ab}^k,\mathsf{Ab}).$$
- Eilenberg and Mac Lane actually use a different notation for the cross-effects, namely
$$T(A_1|\ldots|A_n) \coloneqq cr_nT(A_1,\ldots,A_n).$$
They prove some lemmas showing that the ambiguities that this notation introduces vanish. For example, the notation $T(A_1|A_2|A_3)$ could mean 3 things: you could consider the functor $cr_2T(A_1,-)$ and take its second cross-effect evaluated at (A_2,A_3), you could consider instead $cr_2T(-,A_3)$ and take its second cross-effect evaluated at (A_1,A_2) or you could simply take the third cross-effect of T evaluated at (A_1,A_2,A_3):
$$\begin{gathered} cr_2(cr_2T(A_1,-))(A_2,A_3) \\ cr_2(cr_2T(-,A_3))(A_1,A_2) \\ cr_3T(A_1,A_2,A_3) \end{gathered}$$
They prove that these are all equal (not just isomorphic) as subgroups of $T(A_1 \oplus A_2 \oplus A_3)$. Similarly, for any multireduced functor $T \colon \mathsf{Ab}^k \to \mathsf{Ab}$, the order in which you form cross-effects to obtain, say, $T(A_1|A_2|A_3, B_1|B_2)$ is irrelevant (here, $k = 2$).

Definition 2.3 A functor $T \colon \mathsf{Ab} \to \mathsf{Ab}$ has degree $\leqslant k$ if and only if $cr_{k+1}T = 0$.

Theorem 2.4 *Suppose we have functors $S, T \colon \mathsf{Ab} \to \mathsf{Ab}$ commuting with filtered colimits and a natural transformation $\theta \colon S \to T$. The following are sufficient conditions to deduce that θ is an isomorphism:*

(a) *The induced homomorphism $cr_kS(A_1,\ldots,A_k) \longrightarrow cr_kT(A_1,\ldots,A_k)$ is an isomorphism whenever the A_i are cyclic groups of prime power or infinite order.*

(b) *The functors S, T are both finite-degree and the induced homomorphism $cr_kS(A_1,\ldots,A_k) \longrightarrow cr_kT(A_1,\ldots,A_k)$ is an isomorphism whenever the A_i are cyclic groups of order p^n or ∞ for the same p.*

Eilenberg and Mac Lane use the term *p-cyclic group* to denote cyclic groups of order p^n, for $n \geqslant 0$, or ∞. The point of part (b) is that if S and T have finite degree, one only needs to check the isomorphism when all inputs are p-cyclic groups for the *same* p. Also, it is clearly only necessary to check this for k up to the common degree of S and T.

Part (a) is easy to prove, using the natural decomposition (2.1). First, if A is finitely generated, then it can be written as a sum of p-cyclic groups (for varying p), and naturality of (2.1) plus the assumption in (a) implies that $\theta_A \colon S(A) \to T(A)$ is an isomorphism. Since every abelian group is a (filtered) colimit of its finitely generated subgroups, the result follows from the fact that S and T commute with filtered colimits.

Part (b) is a slight refinement, and follows from part (a) and the following lemma:

Lemma 2.5 *Let $T \colon \mathsf{Ab}^2 \to \mathsf{Ab}$ be a bireduced functor which has finite degree in each argument. If A and B are abelian groups such that there exist coprime integers r, s such that $rA = 0 = sB$, then $T(A,B) = 0$.*

This implies that any cross-effect terms in which you plug in a p-cyclic group and a q-cyclic group, for primes $p \neq q$, automatically vanish.

Application. We now briefly look at how Eilenberg-MacLane apply their general theory of cross-effects to compute

$$H_q(K(\pi, n); \mathbb{Z}),$$

the integral homology of their eponymous spaces. The idea is to consider

$$T = H_q(-, n) = H_q(K(-, n); \mathbb{Z}) \colon \mathsf{Ab} \longrightarrow \mathsf{Ab}$$

as a functor of abelian groups, and first compute its degree d and its cross-effects. The next step is to compute its value on p-cyclic groups, $T(\mathbb{Z}/p^n)$ and $T(\mathbb{Z})$, which they do explicitly. From this one can try to guess that $U \cong T$ for some understood functor U of degree d. To prove this they then construct a natural transformation $U \to T$ inducing isomorphisms

$$U|_{\mathsf{Cyc}_p} \longrightarrow T|_{\mathsf{Cyc}_p} \qquad \text{and} \qquad cr_k U \longrightarrow cr_k T$$

for $2 \leqslant k \leqslant d$, where $\mathsf{Cyc}_p \subset \mathsf{Ab}$ is the subcategory of p-cyclic groups. It appears (from my incomplete reading of their paper), that they always verify that $cr_k U \to cr_k T$ is an isomorphism on the *whole* category Ab^k, and so they do not in fact use the refinement (b) of Theorem 2.4.

The first step is to compute some cross-effects of $H_q(-, n)$. For the rest of this section, we will assume that $n \geqslant 2$. Using the Hurewicz theorem, to see that $H_q(A, n) = 0$ for $q < n$ and that $H_n(A, n) \cong A$, plus the Künneth theorem, we have:

$$\begin{aligned}
H_q(A|B, n) &= 0 \qquad \text{for } q < 2n \\
H_{2n}(A|B, n) &\cong A \otimes B \\
H_{2n+1}(A|B, n) &\cong \mathrm{Tor}(A, B) \\
H_{2n+2}(A|B, n) &\cong (A \otimes H_{n+2}(B, n)) \oplus (H_{n+2}(A, n) \otimes B).
\end{aligned}$$

Since $- \otimes -$ and $\mathrm{Tor}(-, -)$ are bilinear, the functors $H_q(-, n)$ are quadratic for $q = 2n, 2n+1$. When $n \geqslant 3$ the first line above implies that the right-hand side of the fourth line is linear, and so $H_q(-, n)$ is quadratic also for $q = 2n + 2$.

Now define some quadratic functors $\mathsf{Ab} \to \mathsf{Ab}$ as follows:

$$\begin{aligned}
\Lambda_2(A) &= A \otimes A / \langle a \otimes a \rangle \\
\Omega(A) &= \ldots \\
V(A) &= A \otimes (A/2A).
\end{aligned}$$

We omit the definition of $\Omega(A)$, since it is a complicated presentation with many generators and relations; it may be found on page 94 (§13) of [EML54]. These turn out to have the following cross-effects:

$$\begin{aligned}
\Lambda_2(A|B) &\cong A \otimes B \\
\Omega(A|B) &\cong \mathrm{Tor}(A, B) \\
V(A|B) &\cong A \otimes (B/2B) \oplus (A/2A) \otimes B.
\end{aligned}$$

Each of the functors on the right-hand side is linear in each argument, so these functors are quadratic, as claimed. Putting this together, Eilenberg and Mac Lane prove:

Theorem 2.6 ([EML54, Theorems 23.1–4]) *We have the following natural isomorphisms:*

$$\begin{aligned}
H_5(A, 3) &\cong A/2A \\
H_6(A, 3) &\cong {}_2A \oplus \Lambda_2(A) \\
H_7(A, 3) &\cong A/3A \oplus \Omega(A) \\
H_8(A, 3) &\cong {}_3A \oplus (A \otimes A/2A)
\end{aligned}$$

where ${}_pA$ denotes the p-torsion subgroup of A.

Sketch proof. First, note that the functors $A \mapsto A/nA$ and $A \mapsto {}_pA$ are linear. The functors on the first line are both linear, so it suffices to construct a natural transformation $A/2A \to H_5(A,3)$ and check that it is an isomorphism on p-cyclic subgroups. On the remaining three lines, the left-hand and right-hand sides are both quadratic, so it suffices to construct the natural transformation, check that it is an isomorphism on p-cyclic subgroups and that it induces an isomorphism of second cross-effects. That the cross-effects are isomorphic is clear on the second and third lines, and for the fourth line it follows from the calculation

$$H_8(A|B,3) \cong (A \otimes H_5(B,3)) \oplus (H_5(A,3) \otimes B) \cong (A \otimes B/2B) \oplus (A/2A \otimes B),$$

applying the result of the first line. □

Remark 2.7 The calculations of [EML54] in fact cover

$$H_{n+k}(K(\pi,n);\mathbb{Z})$$

for any abelian group π, any $n \geqslant 2$ and any $k \leqslant 5$ — with the single exception of $H_7(K(\pi,2);\mathbb{Z})$.

3 Categories with finite coproducts and Taylor towers

We now generalise the previous setting to the following, for which our reference is [HV11]. Let $\mathcal{C}$ be any category with finite coproducts (denoted $\vee$) and a null (initial and terminal) object 0, and let $\mathcal{D}$ be either the category of groups or any abelian category. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor such that $F(0) = 0$ (reduced).

Since 0 is terminal we have canonical morphisms

$$\begin{tikzcd}[column sep=small] & c_1 \vee c_2 \arrow[dl] \arrow[dr] & \\ c_2 = 0 \vee c_2 & & c_1 \vee 0 = c_1 \end{tikzcd}$$

We can consider the morphism $F(c_1 \vee c_2) \to F(c_1) \times F(c_2)$ whose components are given by $F(-)$ applied to each of the above morphisms. More generally, we have morphisms $F(c_1 \vee \ldots \vee c_n) \to \prod_{i=1}^n F(c_1 \vee \ldots \vee \hat{c}_i \vee \ldots \vee c_n)$ defined in the same way, and morphisms $\prod_{i=1}^n F(c_1 \vee \ldots \vee \hat{c}_i \vee \ldots \vee c_n) \to F(c_1 \vee \ldots \vee c_n)$ defined using the canonical morphisms arising from the fact that 0 is also *initial*.

Definition 3.1 The cross-effects of F are defined recursively by

$$\begin{aligned} cr_1 F &= F \\ cr_2 F(c_1,c_2) &= \ker(F(c_1 \vee c_2) \to F(c_1) \times F(c_2)) \\ cr_n F(c_1,\ldots,c_n) &= cr_2(cr_{n=1}(-,c_3,\ldots,c_n))(c_1,c_2). \end{aligned}$$

Equivalently, we may define the cross-effects in one go by

$$\begin{aligned} cr_n F(c_1,\ldots,c_n) &= \ker\big(F(c_1 \vee \ldots \vee c_n) \to \textstyle\prod_{i=1}^n F(c_1 \vee \ldots \vee \hat{c}_i \vee \ldots \vee c_n)\big) \\ &= \textstyle\bigcap_{i=1}^n \ker\big(F(c_1 \vee \ldots \vee c_n) \to F(c_1 \vee \ldots \vee \hat{c}_i \vee \ldots \vee c_n)\big). \end{aligned}$$

Note also that there is a natural isomorphism

$$cr_n F(c_1,\ldots,c_n) \cong \operatorname{coker}\big(\textstyle\prod_{i=1}^n F(c_1 \vee \ldots \vee \hat{c}_i \vee \ldots \vee c_n) \to F(c_1 \vee \ldots \vee c_n)\big). \tag{3.1}$$

Remark 3.2 The first two definitions use the terminal property of 0 and the cokernel description (3.1) uses the initial property of 0. One can also give a definition by definining certain idempotent endomorphisms in $\mathcal{C}$, then taking their images under T, taking an alternating sum of these, and then taking the image of the resulting endomorphism in $\mathcal{D}$. See §5 for the details of this description. This is a direct generalisation of the definition given by Eilenberg and Mac Lane (see §2) when $\mathcal{C} = \mathcal{D} = \mathsf{Ab}$. In this definition, both initiality and terminality of 0 are used to define the idempotents.

Another remark is that the nth cross-effect was defined in Rosona's talk as the total cofibre of a certain cube, namely

$$\mathcal{P}(\{1,\ldots,n\}) \longrightarrow \mathcal{C} \qquad S \mapsto F(\textstyle\bigvee_{i\in S} c_i) \tag{3.2}$$

where the maps in the cube are the image under F of certain canonical maps coming from the fact that 0 is initial. The cokernel description (3.1) corresponds to the total cofibre of the *restriction* of this cube to its top two layers, i.e. restricting the diagram (3.2) to the subposet of $\mathcal{P}\{1,\ldots,n\}$ consisting of subsets of cardinality n or $n-1$.

As before, the cross-effects $cr_n F\colon \mathcal{C}^n \to \mathcal{D}$ are symmetric and multireduced, and there is a natural decomposition of $F(c_1 \vee \ldots \vee c_n)$ in terms of the first n cross-effects.

Proposition 3.3 ([HV11, Proposition 1.4]) *There is natural tranformation of functors $\mathcal{C}^n \to \mathcal{D}$:*

$$F(c_1 \vee \ldots \vee c_n) \;\cong\; \bigoplus_{i=1}^{n} \bigoplus_{j_1<\cdots<j_i} cr_i F(c_{j_1},\ldots,c_{j_i}).$$

Definition 3.4 We say that F is *polynomial of degree* $\leqslant n$ if $cr_{n+1}F = 0$, and write

$$\mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n} \subseteq \mathsf{Fun}(\mathcal{C},\mathcal{D})$$

for the full subcategory whose objects are polynomial of degree $\leqslant n$.

Remark 3.5 This is a *thick* subcategory, meaning that it is closed under taking quotients, subobjects and extensions.

The Taylor tower. The universal property of the coproduct provides us with a *fold map*

$$c^{\vee n} = c \vee \ldots \vee c \longrightarrow c$$

for any object $c \in \mathcal{C}$, which is the identity on each component. Now apply F to this morphism, and recall that $cr_n F(c_1,\ldots,c_n)$ was defined as a subobject of $F(c_1,\ldots,c_n)$, so we get a composite

$$cr_n F(c,\ldots,c) \hookrightarrow F(c^{\vee n}) \longrightarrow F(c)$$

in $\mathcal{D}$. This construction is functorial in c, in other words it constitutes a natural transformation from $cr_n F \circ \Delta$ to F, where $\Delta\colon \mathcal{C} \to \mathcal{C}^n$ is the diagonal. So we have a morphism $cr_n F \circ \Delta \to F$ in the abelian category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$, and we define

$$p_n F \coloneqq \mathrm{coker}(cr_n F \circ \Delta \to F) \in \mathsf{Fun}(\mathcal{C},\mathcal{D}).$$

This construction is in turn functorial in F, so we have an endofunctor $\mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Fun}(\mathcal{C},\mathcal{D})$.

Lemma 3.6 ([HV11, Proposition 1.10]) *For any $F \in \mathsf{Fun}(\mathcal{C},\mathcal{D})$, the functor $p_n F$ is polynomial of degree $\leqslant n$. In other words the above functor factors through a functor*

$$p_n\colon \mathsf{Fun}(\mathcal{C},\mathcal{D}) \longrightarrow \mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n}.$$

Moreover, p_n is left adjoint to the inclusion functor $u_n\colon \mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n} \longrightarrow \mathsf{Fun}(\mathcal{C},\mathcal{D})$, and the unit of this adjunction $t_n\colon \mathrm{id} \Rightarrow u_n \circ p_n$ is an isomorphism on objects in $\mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n} \subset \mathsf{Fun}(\mathcal{C},\mathcal{D})$.

Rephrasing, we have an adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{C},\mathcal{D}) \arrow[r, shift left, "p_n"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n} \arrow[l, shift left, "u_n"] \end{tikzcd}$$

and the unit $t_n\colon F \to p_nF$ is an isomorphism if F is already polynomial of degree $\leqslant n$. Hence in particular $\mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n}$ is a *reflective* subcategory of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ (this just means that the inclusion admits some left adjoint). But this lemma tells us more: we don't just know that some left adjoint exists, we have an explicit description of it involving the nth cross-effect.

From the basic properties of adjunctions we have the following, where we abuse notation slightly by not writing the inclusion functor u_n explicitly. Each morphism $F \to G$ in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$, with G polynomial of degree $\leqslant n$, factors uniquely as $F \to p_nF \to G$, where the first map is the unit t_n. In other words, for a fixed F, the map $t_n\colon F \to p_nF$ is the inital morphism from F to a functor of degree $\leqslant n$ (in fancy language, it's the initial object in the comma category $F \downarrow \mathsf{Fun}(\mathcal{C},\mathcal{D})_{\leqslant n}$).

The functor $p_{n-1}F$ is polynomial of degree $\leqslant n-1$, so in particular polynomial of degree $\leqslant n$. Thus, by the previous paragraph, the unit $t_{n-1}\colon F \to p_{n-1}F$ must factor uniquely through the unit $t_n\colon F \to p_nF$:

$$F \longrightarrow p_nF \longrightarrow p_{n-1}F.$$

Denote the unique morphism $p_nF \to p_{n-1}F$ above by q_n. Putting this all together, we have the Taylor tower associated to F:

$$\begin{tikzcd}
F \arrow[d, "t_1"'] \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[drrrr, "t_5"] & & & & & \\
0 = p_1F & p_2F \arrow[l, "q_2"] & p_3F \arrow[l, "q_3"] & p_4F \arrow[l, "q_4"] & p_5F \arrow[l, "q_5"] & \cdots \arrow[l, "q_6"]
\end{tikzcd}$$

Aside: a concrete description of the maps between the levels. Write $\mathrm{id}^{\vee n-1} \vee \mathrm{fold}$ for the map $c^{\vee n+1} \to c^{\vee n}$ that folds together the last two copies of c. This induces a map $F(c^{\vee n+1}) \to F(c^{\vee n})$, and note that this in turn restricts to a morphism between the subobjects $cr_{n+1}F(c,\ldots,c)$ and $cr_nF(c,\ldots,c)$. We therefore have a commutative diagram

$$\begin{tikzcd}
cr_{n+1}F(c,\ldots,c) \arrow[r, hook] \arrow[d] & F(c^{\vee n+1}) \arrow[r] \arrow[d, "F(\mathrm{id}^{\vee n-1} \vee \mathrm{fold})"'] & F(c) \arrow[r] \arrow[d, "\mathrm{id}"] & p_{n+1}F(c) \\
cr_nF(c,\ldots,c) \arrow[r, hook] & F(c^{\vee n}) \arrow[r] & F(c) \arrow[r] & p_nF(c).
\end{tikzcd}$$

We therefore get an induced map $p_{n+1}F(c) \to p_nF(c)$ between the cokernels, which is precisely the natural transformation q_{n+1} on the object c.

Exercise: check that this claim is true! It suffices to check that (a) the map $F(\mathrm{id}^{\vee n-1} \vee \mathrm{fold})$ really does restrict to a map $cr_{n+1}F(c,\ldots,c) \to cr_nF(c,\ldots,c)$ and (b) we have $q_{n+1} \circ t_{n+1} = t_n$ when q_{n+1} is defined as above.

A criterion for a natural transformation between polynomial functors. Hartl and Vespa note (see [HV11, Proposition 1.17]) that when $\mathcal{C}$ is generated under the coproduct by a single object c, then one can check that natural transformations between functors are isomorphisms by checking this just on cross-effects applied to tuples $(c,\ldots,c)$. More precisely:

Proposition 3.7 *Suppose that $\mathcal{C} = \{0, c, c^{\vee 2}, \ldots, c^{\vee n}, \ldots\}$, $\mathcal{D}$ is an abelian category and F, G are polynomial functors $\mathcal{C} \to \mathcal{D}$ of degree $\leqslant n$. Then a natural transformation $\phi\colon F \Rightarrow G$ is an isomorphism provided that the morphisms*

$$cr_k(\phi)\colon cr_kF(c,\ldots,c) \longrightarrow cr_kG(c,\ldots,c)$$

in $\mathcal{D}$ are isomorphisms for $k \leqslant n$.

Remark 3.8 Some analogies and related facts:

- This is analogous to Theorem 10.4 of [EML54] (see part (b) of Theorem 2.4), where it sufficed to check the maps of cross-effects on tuples of p-cyclic groups for each prime p. The difference is just that the category Ab is generated under finite coproducts (and filtered colimits) by infinitely many objects – the p-cyclic groups – rather than a single object.
- The analogous fact about polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$ is Corollary 1.2. Here, a polynomial is determined uniquely by the values of the cross-effects $cr_k f$ on the real number 1 (or indeed any other choice of non-zero real number) for $k \leqslant \deg(f)$.
- There is a much more general result of this kind in Theorem 2.11 of [JM03], where $\mathcal{C}$ is "generated" by c in a less strict way (allowing also "resolutions along c").

4 Monoidal categories whose unit is null

We now briefly note that a lot of the previous section generalises to the following setting, considered in [HPV12]. Let $\mathcal{M}$ be a monoidal category whose unit $I = 0$ is null, i.e. both initial and terminal. Let $\mathcal{D}$ be an abelian category and let $F\colon \mathcal{M} \to \mathcal{D}$ be a reduced functor (meaning as always that $F(0) = 0$). Note that any category $\mathcal{M}$ with finite coproducts and a null object is of this form, with monoidal product given by the categorical coproduct.

In this more general setting, everything in the previous section, up to and including Remark 3.5, goes through exactly as before, replacing $\vee$ with the monoidal product $\otimes$. So we still have all the descriptions of the cross-effects, the decomposition (Proposition 3.3) and a thick, full subcategory $\mathsf{Fun}(\mathcal{M},\mathcal{D})_{\leqslant n} \subset \mathsf{Fun}(\mathcal{M},\mathcal{D})$ of functors which are *polynomial of degree* $\leqslant n$. However, there is no fold map $c \otimes \cdots \otimes c \to c$ in this general setting, and so the construction of the Taylor tower in the previous section does not generalise.

Question: in particular, we cannot explicitly construct a left adjoint to the inclusion $\mathsf{Fun}(\mathcal{M},\mathcal{D})_{\leqslant n} \hookrightarrow \mathsf{Fun}(\mathcal{M},\mathcal{D})$ in this generality, but does one exist for abstract reasons? For example, as Ilan pointed out, if we know that $\mathsf{Fun}(\mathcal{M},\mathcal{D})$ is locally presentable and $\mathsf{Fun}(\mathcal{M},\mathcal{D})_{\leqslant n}$ is closed under limits, then Vopěnka's principle non-constructively implies that $\mathsf{Fun}(\mathcal{M},\mathcal{D})_{\leqslant n}$ admits a left adjoint.

An example of an interesting category that fits into this framework, but not into the previous section, is the category $\mathrm{FI}\sharp$ of finite sets and partially-defined injections. A partially-defined injection is a choice of subset together with an injection defined on that subset, and composition is a special case of composition in the category of relations, i.e., the composite $f \circ g$ is defined on an element x if and only if g is defined on x and f is defined on $g(x)$.

This category appears in [CEF12] in relation to representation stability (it also appears in [DV13], where it is denoted $\widetilde{\Theta}$). The subcategory $\mathrm{FI} \subset \mathrm{FI}\sharp$ of finite sets and (everywhere-defined) injections is a monoidal subcategory whose unit is still initial, but no longer terminal. The *FI-modules* of [CEF12] are functors from this category to R-mod for some ring R, and therefore belong to the framework of the next section.

The constructions available at different levels of generality can be summarised in the following table (which also includes a preview of the next section).

			cross-effects	c-e decomp$^{\mathrm{n}}$	Taylor tower
§3	[HV11]	finite coproducts, null object	✓	✓	✓
§4	[HPV12]	monoidal, unit is null	✓	✓	—
§5	[DV13]	monoidal, unit is initial	(✓)	—	—

The (✓) indicates that one can define the cross-effects as in the two previous cases, but at this level of generality they do not behave as well as one wants (they don't commute with limits). The method of dealing with this issue is the main topic of the next section.

5 Monoidal categories whose unit is initial: inductive degree and examples related to the homology of the Torelli groups

In this final section, we consider a monoidal category $\mathcal{M}$ whose unit object I is initial, but not necessarily terminal. As before, $\mathcal{D}$ denotes an abelian category, and functors $F\colon \mathcal{M} \to \mathcal{D}$ are assumed to be reduced, meaning that $F(I) = 0$. The reference for this section is [DV13]. I covered this in a bit less detail in the talk than is written here.

Example 5.1 First, we give some motivating examples.

- The category FI of finite sets and injections, with monoidal product given by disjoint union, and initial object the empty set. This appears in [CEF12], as well as [SS14, §7] and [DV13] (where it is denoted Θ).
- If $\mathcal{A}$ is any additive category, there is a category $S(\mathcal{A})$ with the same objects, and where a morphism $a \to b$ is a morphism $f\colon a \to b$ in $\mathcal{A}$ together with a choice of left-inverse $s\colon b \to a$ (i.e. $sf = \mathrm{id}_a$). The fact that $\mathcal{A}$ has finite biproducts gives it the structure of a monoidal category whose unit is null, and gives $S(\mathcal{A})$ the structure of a monoidal category whose unit is initial, but *not* terminal.
- In particular, let $\mathsf{ab} \subset \mathsf{Ab}$ denote the full subcategory of finite-rank free abelian groups. This is an additive category, so we may form $S(\mathsf{ab})$ as in the previous example. An alternative description of a morphism in $\mathrm{Hom}_{S(\mathsf{ab})}(\mathbb{Z}^m, \mathbb{Z}^n)$ is: an injective homomorphism $i\colon \mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$, together with a choice of subgroup $G \leqslant \mathbb{Z}^n$ such that $\mathbb{Z}^n = G \oplus i(\mathbb{Z}^m)$.
- There is a non-abelian analogue of this category, denoted $\mathcal{G}$. Its objects are the finite-rank free groups, and a morphism from F_m to F_n is an injective homomorphism $i\colon F_m \hookrightarrow F_n$, together with a choice of subgroup $G \leqslant F_n$ such that $F_n = G * i(F_m)$, where $*$ denotes the free product of groups. This category is not of the form $S(\mathcal{A})$ for an additive category $\mathcal{A}$, but it does have a monoidal product, given by the free product of groups, and its initial object is the trivial group. Abelianisation induces a canonical functor $ab\colon \mathcal{G} \to S(\mathsf{ab})$.
- A more general construction of monoidal categories with initial unit is given in [DV13, §3.2]. Given an additive category $\mathcal{A}$ and some extra data (a "duality functor" and an additional 4-way choice), they define the associated *category of hermitian objects* $H(\mathcal{A})$. This recovers some of the above examples: there is a natural choice of the additional structure on $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ such that $H(\mathcal{A}^{\mathrm{op}} \times \mathcal{A}) \cong S(\mathcal{A})$.

First, we will give some more details of the case where the unit object I is null, and give an alternative, inductive definition of the *degree* of a polynomial functor $\mathcal{M} \to \mathcal{D}$. After that, we will see what goes wrong when I is not terminal, and briefly sketch the solution to this problem given in [DV13], which involves generalising the inductive definition of degree, rather than the cross-effects. Finally, in §5.3, we mention some interesting functors $\mathcal{M} \to \mathcal{D}$ to which this theory may be applied.

5.1 When the unit is null

In the previous section we noted that the definition of cross-effects given in Definition 3.1 generalises directly to this situation, replacing $\vee$ with the monoidal product $\otimes$. In particular, the description (3.1) of the cross-effects as a cokernel holds in this generality – for a reducted functor $F\colon \mathcal{M} \to \mathcal{D}$ we have:

$$cr_n F(c_1, \ldots, c_n) \cong \mathrm{coker}\big(\textstyle\prod_{i=1}^n F(c_1 \otimes \ldots \otimes \hat{c}_i \otimes \ldots \vee c_n) \to F(c_1 \otimes \ldots \otimes c_n)\big). \tag{5.1}$$

Another description is a generalisation of the one we saw in §2, given by Eilenberg and Mac Lane. For objects $c_1, \ldots, c_n \in \mathcal{M}$ and a subset $S \subseteq \{1, \ldots, n\}$, define idempotents

$$e_S(c_1, \ldots, c_n) = \bigoplus \left\{ \begin{smallmatrix} \mathrm{id} & i \in S \\ 0 & i \notin S \end{smallmatrix} \right\} \colon c_1 \otimes \cdots \otimes c_n \longrightarrow c_1 \otimes \cdots \otimes c_n.$$

Applying F and taking an alternating sum, we obtain an endomorphism

$$f_F(c_1, \ldots, c_n) \coloneqq \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{n-|S|} F(e_S(c_1, \ldots, c_n)) \colon F(c_1 \otimes \cdots \otimes c_n) \longrightarrow F(c_1 \otimes \cdots \otimes c_n)$$

in $\mathcal{D}$. Then we have:

$$cr_n F(c_1, \dots, c_n) = \operatorname{im}(f_F(c_1, \dots, c_n)). \tag{5.2}$$

A basic property of the cross-effect functors is the following:

Lemma 5.2 ([DV13, Proposition 1.7]) *The functors* $cr_n \colon \mathsf{Fun}(\mathcal{M}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{M}^n, \mathcal{D})$ *commute with limits and colimits.*

As before, a functor $F \colon \mathcal{M} \to \mathcal{D}$ is *polynomial of degree* $\leqslant n$ if $cr_{n+1}F = 0$. We now give an alternative, inductive definition of the degree. For a fixed object $c \in \mathcal{M}$, we define certain endofunctors of $\mathsf{Fun}(\mathcal{M}, \mathcal{D})$. The first is the *translation functor*

$$\tau_c \colon \mathsf{Fun}(\mathcal{M}, \mathcal{D}) \longrightarrow \mathsf{Fun}(\mathcal{M}, \mathcal{D}),$$

which takes $F \colon \mathcal{M} \to \mathcal{D}$ to the composite $\mathcal{M} \xrightarrow{c \otimes -} \mathcal{M} \xrightarrow{F} \mathcal{D}$. Since the unit I is *initial*, we have, for each $F \in \mathsf{Fun}(\mathcal{M}, \mathcal{D})$ and $d \in \mathcal{M}$, a morphism $F(d) = F(I \otimes d) \to F(c \otimes d)$. These assemble to form a natural transformation $\mathrm{id} \Rightarrow \tau_c$ between endofunctors of $\mathsf{Fun}(\mathcal{M}, \mathcal{D})$. More generally, we note that the construction $c \mapsto \tau_c \colon \mathcal{M} \to \mathsf{End}(\mathsf{Fun}(\mathcal{M}, \mathcal{D}))$ is functorial, and the unique morphism $I \to c$ therefore induces a natural transformation $\mathrm{id} = \tau_I \Rightarrow \tau_c$, which is the one described above.

Now, since we moreover have a zero object, we also have a zero morphism $o_c \colon c \to c$ for each $c \in \mathcal{M}$. Applying τ to this morphism, we obtain an endomorphism $\tau_{o_c} \colon \tau_c \to \tau_c$ in $\mathsf{End}(\mathsf{Fun}(\mathcal{M}, \mathcal{D}))$. This is an abelian category, so we may define $\delta_c = \operatorname{im}(\mathrm{id}_{\tau_c} - \tau_{o_c})$ to obtain the *difference functor*

$$\delta_c \colon \mathsf{Fun}(\mathcal{M}, \mathcal{D}) \longrightarrow \mathsf{Fun}(\mathcal{M}, \mathcal{D}).$$

By construction, δ_c is the cokernel of the natural transformation $\mathrm{id} \Rightarrow \tau_c$. On the other hand, the kernel is trivial. This is due to the fact that I is *terminal*, as well as initial: the unique morphism $I \to c$ is split-injective, and therefore so are the morphisms $F(d) \to F(c \otimes d)$ which form the natural transformation $\mathrm{id} \Rightarrow \tau_c$. We therefore have a short exact sequence

$$0 \to \mathrm{id} \longrightarrow \tau_c \longrightarrow \delta_c \to 0$$

in $\mathsf{End}(\mathsf{Fun}(\mathcal{M}, \mathcal{D}))$. Moreover, δ_c was defined as a subobject of τ_c (in fact, as the image of a certain morphism $\tau_c \to \tau_c$), and the inclusion is a splitting for the above short exact sequence.

Definition 5.3 We define the *inductive degree* of $F \colon \mathcal{M} \to \mathcal{D}$ as follows:

$$\begin{array}{ll} \deg'(F) = -1 & \text{if } F = 0 \\ \deg'(F) \leqslant n & \text{if } \deg'(\delta_c F) \leqslant n-1 \text{ for all } c \in \mathcal{M} \\ \deg'(F) = \infty & \text{otherwise.} \end{array}$$

Proposition 5.4 ([DV13, Proposition 1.12]) *For all* $F \colon \mathcal{M} \to \mathcal{D}$, $\deg(F) = \deg'(F)$. *In particular,* F *is polynomial if and only if its inductive degree is finite.*

5.2 When the unit is initial

We now briefly discuss what happens when we remove the assumption that I is terminal.

Cross-effects. Since we have no null object, we cannot define the idempotents required to describe the cross-effects as in (5.2). The description (5.1) still works in this setting though, since it assumes only that I is initial. However, there is no cross-effect decomposition analogous to Proposition 3.3.

Difference functors. The discussion above Definition 5.3 goes through, except that the natural transformation $\mathrm{id} \Rightarrow \tau_c$ may have non-trivial kernel, and we cannot define its cokernel δ_c as we did previously. Instead, we may simply *define* δ_c to be the cokernel of the natural transformation, and similarly define κ_c to be its kernel. We therefore have an exact sequence

$$0 \to \kappa_c \longrightarrow \mathrm{id} \longrightarrow \tau_c \longrightarrow \delta_c \to 0$$

in $\mathsf{End}(\mathsf{Fun}(\mathcal{M}, \mathcal{D}))$.

Degree. We may define $\deg(F)$, as before, as the smallest n such that $cr_{n+1}F = 0$. We may also define $\deg'(F)$, as in Definition 5.3, using the difference functor δ_c — and again $\deg(F) = \deg'(F)$. A functor $F\colon \mathcal{M} \to \mathcal{D}$ is called *strong polynomial of degree* $\leqslant n$ in [DV13] if its degree is at most n (and this notion of degree is correspondingly called its *strong degree*). The full subcategory of strong polynomial functors of degree $\leqslant n$ is denoted $\mathsf{Fun}(\mathcal{M}, \mathcal{D})_{\leqslant n}$.

The problem. However, this notion of degree does behave as well as one would like. In contrast to Lemma 5.2, the cross-effects $cr_n\colon \mathsf{Fun}(\mathcal{M}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{M}^n, \mathcal{D})$ do not commute with limits. The difference functors $\delta_c\colon \mathsf{Fun}(\mathcal{M}, \mathcal{D}) \to \mathsf{Fun}(\mathcal{M}, \mathcal{D})$ also do not commute with limits, and the subcategory $\mathsf{Fun}(\mathcal{M}, \mathcal{D})_{\leqslant n} \subset \mathsf{Fun}(\mathcal{M}, \mathcal{D})$ is not closed under limits.

How this is fixed. The idea is to kill the kernels κ_c by passing to a quotient of the category $\mathsf{Fun}(\mathcal{M}, \mathcal{D})$. A technical caveat is that we now assume that $\mathcal{D}$ is not just abelian, but a Grothendieck category. This still includes the example Ab, and more generally R-Mod for any ring R. Define $\mathsf{Sn}(\mathcal{M}, \mathcal{D})$ to be the full subcategory of $\mathsf{Fun}(\mathcal{M}, \mathcal{D})$ on functors F such that

$$\sum_{c \in \mathcal{M}} \kappa_c F = F.$$

This subcategory is thick (cf. Remark 3.5) and closed under coproducts ([DV13, Proposition 1.24]), so it is a *localising subcategory*: we can form the quotient category $\mathsf{St}(\mathcal{M}, \mathcal{D}) \coloneqq \mathsf{Fun}(\mathcal{M}, \mathcal{D})/\mathsf{Sn}(\mathcal{M}, \mathcal{D})$, which is again a Grothendieck category, and the quotient functor $\mathsf{Fun}(\mathcal{M}, \mathcal{D}) \to \mathsf{St}(\mathcal{M}, \mathcal{D})$ has a right adjoint. The notation "Sn" stands for "stably null" and the "St" stands for "stable". The endofunctors τ_c and δ_c of $\mathsf{Fun}(\mathcal{M}, \mathcal{D})$ descend to endofunctors of $\mathsf{St}(\mathcal{M}, \mathcal{D})$, and we now have a short exact sequence

$$0 \to \mathrm{id} \longrightarrow \tau_c \longrightarrow \delta_c \to 0$$

in $\mathsf{End}(\mathsf{St}(\mathcal{M}, \mathcal{D}))$. It now turns out [DV13, Corollaire 1.30] that τ_c commutes with limits, and for any diagram ϕ in $\mathsf{St}(\mathcal{M}, \mathcal{D})$, the canonical morphism $\delta_c(\lim \phi) \to \lim(\delta_c \circ \phi)$ is a monomorphism. We now apply the *inductive* definition of degree (as in Definition 5.3) to define $\deg(F)$ for objects $F \in \mathsf{St}(\mathcal{M}, \mathcal{D})$. A *polynomial functor of degree* $\leqslant n$ is then defined to be an object of $\mathsf{St}(\mathcal{M}, \mathcal{D})$ (or a representative of it) whose (inductive) degree is at most n, and $\mathsf{St}(\mathcal{M}, \mathcal{D})_{\leqslant n}$ denotes the full subcategory of such objects. It then turns out that:

Proposition 5.5 ([DV13, Proposition 1.36]) *The subcategory* $\mathsf{St}(\mathcal{M}, \mathcal{D})_{\leqslant n}$ *is closed under limits.*

5.3 Examples of functors

Having explained some of the theory of polynomial functors out of a monoidal category $\mathcal{M}$ with initial unit I, we now give some examples of interesting functors $\mathcal{M} \to \mathsf{Ab}$ to which this theory may be applied. In these examples, $\mathcal{M}$ will always be one of the categories described in Example 5.1 at the beginning of this section.

FI-modules. Let $\mathcal{M}$ be the category $\Theta = \mathrm{FI}$. Functors $\Theta \to \mathsf{Ab}$ are then called *FI-modules* in [CEF12]. For example, if one has a functor $\mathrm{FI}^{\mathrm{op}} \to \mathsf{hTop}$ to the homotopy category of spaces, then its composition with, say, rational cohomology $H^i(-;\mathbb{Q})\colon \mathsf{hTop}^{\mathrm{op}} \to \mathbb{Q}\text{-Mod} \subset \mathsf{Ab}$ is an FI-module. The notion of *finite generation* of an FI-module introduced in [CEF12] is equivalent to being a strong polynomial functor in the sense of [DV13].

For example, given any space M, there is a functor $\mathrm{Conf}(M)\colon \mathrm{FI}^{\mathrm{op}} \to \mathsf{hTop}$ taking a finite set S to the subspace $\mathrm{Conf}_S(M) \subseteq M^S$ of tuples of pairwise-distinct points in M – in other words, the space of ordered configurations in M with $|S|$ points. If M is a smooth, connected, orientable manifold of dimension at least 2 and with finitely-generated rational cohomology, the composite functor $H^i(\mathrm{Conf}(M);\mathbb{Q})\colon \mathrm{FI} \to \mathbb{Q}\text{-Mod}$ is a finitely-generated FI-module, i.e., a strong polynomial functor (see [CEF12, Theorem 6.2.1]).

Torelli subgroups of automorphism groups of free groups. Any automorphism of a group G acts on its abelianisation G^{ab}, so there is a homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(G^{\mathrm{ab}})$. In particular, when $G = \mathbb{Z}$, we have a homomorphism $\mathrm{Aut}(F_n) \to GL_n(\mathbb{Z})$. This is surjective, and we denote its kernel by IA_n – this is the *Torelli subgroup* of $\mathrm{Aut}(F_n)$:

$$0 \to IA_n \longrightarrow \mathrm{Aut}(F_n) \longrightarrow GL_n(\mathbb{Z}) \to 0.$$

Except in degree 1 and partially in degree 2 (see [Pet05] and [DP14]), the homology of IA_n is very mysterious. The assignment $n \mapsto H_i(IA_n)$ for a fixed degree $i \geqslant 0$ extends to a functor $\mathcal{G} \to \mathsf{Ab}$, where $\mathcal{G}$ is the category of finite-rank free groups described in Example 5.1. This functor factors through the abelianisation functor $ab\colon \mathcal{G} \to S(\mathsf{ab})$, so we may instead study the functor

$$H_i(IA)\colon S(\mathsf{ab}) \longrightarrow \mathsf{Ab}$$

through which it factors. When $i = 1$, this is well-understood, since there is a natural isomorphism $(IA_n)^{\mathrm{ab}} \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}^n, \Lambda^2(\mathbb{Z}^n))$, which implies that the functor $H_1(IA)$ has degree 3, both as an object of $\mathsf{Fun}(S(\mathsf{ab}), \mathsf{Ab})$ (so it has *strong* degree 3) and as an object of $\mathsf{St}(S(\mathsf{ab}), \mathsf{Ab})$. For $i \geqslant 2$, the degree of $H_i(IA)$ – including whether or not it is finite – is not currently known, although Djament and Vespa say that in future work they will show that for each i, the functor

$$H_i(IA) \in \mathsf{St}(S(\mathsf{ab}), \mathsf{Ab})$$

is polynomial, i.e., has finite degree, inspired by related results proved in [Put12] and [Chu+12]. However, determining the precise degree of these functors seems, they say, to be a "particularly delicate" problem.

Another interesting functor $S(\mathsf{ab}) \to \mathsf{Ab}$ mentioned in [DV13] is given on objects by

$$n \;\mapsto\; i\text{th quotient of the lower central series of } IA_n.$$

Congruence subgroups. Let I be a ring without unit. We can formally adjoin a unit to obtain a ring $I \oplus \mathbb{Z}$ with unit. There is then a surjective homomorphism between their automorphism groups $GL_n(I \oplus \mathbb{Z}) \to GL_n(\mathbb{Z})$, and we denote its kernel by $GL_n(I)$ – this is the degree-n *congruence group* of I (also called the *congruence subgroup* of the general linear group $GL_n(I \oplus \mathbb{Z})$):

$$0 \to GL_n(I) \longrightarrow GL_n(I \oplus \mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}) \to 0.$$

For any fixed homological degree $i \geqslant 0$, the assignment $n \mapsto H_i(GL_n(I))$ extends to a functor

$$H_i(GL(I))\colon S(\mathsf{ab}) \longrightarrow \mathsf{Ab}.$$

In §1.2.4 of [DV13], they mention that the study of these functors is related to the question of excision in algebraic K-theory: they state that the ring-without-unit I satisfies excision in algebraic K-theory if and only if each of the functors $H_i(GL(I))$ has degree 0 (as an object in $\mathsf{St}(S(\mathsf{ab}), \mathsf{Ab})$).

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